Noncommutative Updating of Perceived Self-Orientation in Three Dimensions

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Glasauer S, Brandt T. Noncommutative updating of perceived self-orientation in three dimensions. J Neurophysiol 97: 2958–2964, 2007. First published February 7, 2007; doi:10.1152/jn.00655.2006. After whole body rotations around an earth-vertical axis in darkness, subjects can indicate their orientation in space with respect to their initial orientation reasonably well. This is possible because the brain is able to mathematically integrate self-velocity information provided by the vestibular system to obtain self-orientation, a process called path integration. For rotations around multiple axes, however, computations are more demanding to accurately update self-orientation with respect to space. In such a case, simple integration is no longer sufficient because of the noncommutativity of rotations. We investigated whether such updating is possible after three-dimensional whole body rotations and whether the noncommutativity of three-dimensional rotations is taken into account. The ability of ten subjects to indicate their spatial orientation in the earth-horizontal plane was tested after different rotational paths from upright to supine positions. Initial and final orientations of the subjects were the same in all cases, but the paths taken were different, and so were the angular velocities sensed by the vestibular system. The results show that seven of the ten subjects could consistently indicate their final orientation within the earth-horizontal plane. Thus perceived final orientation was independent of the path taken, i.e., the noncommutativity of rotations was taken into account.

INTRODUCTION

Vestibular input signaling angular self-velocity is sufficient to notice changes of self-orientation in space (Mach 1875). Previous experiments showed that subjects can estimate their orientation after short whole body rotations around the earth-vertical axis in complete darkness (e.g., Guedry 1974; Israel et al. 1995; Jürgens et al. 1999; Marlinsky 1999; Mergner et al. 1991; Mittelstaedt and Mittelstaedt 1997). During this task, the only available information about movement in space is the angular self-velocity sensed by the vestibular system. Consequently, estimating self-orientation in space requires that the angular velocity cue from the vestibular system is mathematically integrated to yield self-orientation. This process is part of an ability called path integration (Mittelstaedt and Mittelstaedt 1980) or inertial idiothesis (Mittelstaedt and Glasauer 1991). Path integration provides means to update knowledge of one’s spatial position and orientation (Klatzky et al. 1998).

If, however, the rotation involves more than a single axis—e.g., first a rotation around the subject’s yaw axis and subsequently a rotation around the pitch axis—a simple integration of angular velocity no longer suffices because of the kinematics of three-dimensional (3D) rotations. Instead, to accurately compute self-orientation in space from angular velocity signals, the instantaneous orientation has to be taken into account because rotations do not commute. In other words, the final orientation depends on the order of the applied rotations.

It has been shown that the brain accounts for the noncommutativity of rotations when the vestibuloocular reflex stabilizes gaze during whole body rotations (Tweed et al. 1999) or during the remapping of visual targets during eye or head movements (Crawford et al. 2004; Medendorp et al. 2002; Smith and Crawford 2001). It is not known at present whether this also holds for the updating of self-orientation. A straightforward way to test this hypothesis would be to submit subjects to the same passive whole body rotations, but in different order, thus yielding different final orientations. This approach was used before for the vestibuloocular reflex (Tweed et al. 1999).

Here we used an alternative approach that is equally well suited to test whether noncommutativity is taken into account by the updating process: if different rotations starting at the same initial orientation and yielding the same final orientation are applied, any commutative operation should result in differences in perceived final self-orientation. This approach can also be regarded as a test of whether estimation of self-orientation is path independent. The APPENDIX (Noncommutativity and path independence) shows that our approach can indeed be used as an equivalent test of whether the path-integration process was aware of the noncommutativity of rotations.

On earth, rotations around more than one axis necessarily involve changes with respect to gravity. The straightforward approach of using the same rotations in different order would thus lead to different final orientations with respect to gravity, making it possible to infer the criterion of inequality of final orientations exclusively from the final graviceptive cues. The approach used here avoids this possibility: using different sequences of rotation, subjects were rotated from the same initial orientation (upright; see Fig. 1, left) to various supine orientations, which were identical with respect to gravity (Fig. 1, right). Therefore gravity cues in the final position could not be used to distinguish orientations. However, final subject orientations differed with respect to the angle α within the earth-horizontal plane (e.g., different sequences of rotation

APPENDIX (Noncommutativity and path independence)

Path integration usually refers to the temporal integration of self-velocity to yield self-displacement. However, in accordance with the original definition (Mittelstaedt and Mittelstaedt 1980), we suggest that any process is termed path integration, if it yields self-displacement from idiothetic cues such as self-velocity (temporal integration) or instantaneous displacement (spatial integration).

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from upright facing north to supine with the head pointing either west or east). Subjects were then instructed to indicate their remembered initial orientation in the earth-horizontal plane. We hypothesized that if the required path integration process takes into account the noncommutativity of rotations, the same final orientation should correctly be perceived as being equal despite different paths taken to get there, but different final orientations should be distinguished despite equal graviceptive cues.

Preliminary results were reported earlier in abstract form (Glasauer et al. 1996).

METHODS

Experiments

Ten healthy subjects (two females, eight males, ages 25–41 yr) without a history of vestibular or ocular motor problems gave their informed consent to participate in the present experiment, which conformed to the standards set by the Declaration of Helsinki. Subjects sat upright in a two-axis rotating chair (SEGA), initially always facing the same space-fixed direction in the horizontal plane (Fig. 1, left, defined as $\alpha = 0^\circ$). During the experiment, subjects were rotated in complete darkness to six supine orientations (Fig. 1, right). The supine orientations differed by the final orientation $\alpha$ in the horizontal plane, i.e., by the angle between the initial facing direction and the final longitudinal body axis. The final angles $\alpha$ with respect to the initial orientation were $\pm 80^\circ$, $\pm 90^\circ$, and $\pm 100^\circ$. The two-axis rotating chair consists of a gimbal system with the inner subject-fixed axis parallel to the subject’s nasooccipital roll axis and the outer earth-fixed axis is earth-horizontal (Fig. 1). Subjects, secured by safety belts and a head restraint, sat within a cabin that was closed throughout the experiment to shield subjects from airflow and any residual external light. Additionally, the laboratory was darkened throughout the experiment. To mask external sounds, subjects wore headphones, which delivered white noise.

To test for path independence, three different rotations were performed to bring subjects from the initial orientation into the final orientations (Fig. 2): 1) pitch-roll: the subject was first pitched back by $90^\circ$ into the supine position and then rotated around the nasooccipital axis by the angle $\alpha$ ($\pm 80^\circ$, $\pm 90^\circ$, and $\pm 100^\circ$) to reach the final orientation; 2) roll-yaw: the subject was first rolled around the angle $\alpha$ ($\pm 80^\circ$, $\pm 90^\circ$, and $\pm 100^\circ$) and subsequently rotated around the earth-horizontal axis by $+90^\circ$ or $-90^\circ$ to reach the supine position (for $\alpha = \pm 90^\circ$, this is a pure yaw rotation); 3) combined: the rotation was performed by rotating both axes of the chair at approximately the same time; this yielded an oblique time-varying axis of rotation resulting from the different peak velocities of the two axes (inner axis peak velocity $100^\circ$/s, outer axis $60^\circ$/s) and final orientations equal to those of pitch-roll and roll-yaw. All rotations were done with trapezoidal velocity profiles (see Fig. 2). Rotations were performed in randomized order.

Figure 2 shows subject positions (as insets) for the three rotations together with the angular velocity stimuli in egocentric coordinates, i.e., the angular velocity as sensed by the vestibular system. Each subject underwent $6 \times 3 = 18$ rotations in randomized order.

Before each rotation, subjects were asked to remember their orientation in the earth-horizontal plane (see Fig. 1, left), i.e., their heading or straight-ahead direction (e.g., looking north while being in the initial position). After each rotation (duration <5 s; see Fig. 2) the supine subjects had to indicate their self-orientation in space by rotating an indicator, a computer-controlled luminous line polarized by an arrowhead and displayed on a dark TV screen, around the nasooccipital axis by remote control. The TV screen was mounted inside the chair’s cabin in front of the subject, i.e., the screen was always in the same position with respect to the subject. The edges of the screen were masked by a blackboard with a circular hole, and the luminance of the screen was dimmed so that the interior of the cabin was not visible. The subjects were instructed to rotate the line within 30 s so that it pointed along the remembered initial orientation, as if the indicator were a compass needle (i.e., the indicator should have the same orientation in space as the subjects’ straight-ahead before the rotation, e.g., when the initial heading was north, the correct response would have been to rotate the indicator so that it points north). After 30 s, the chair was quickly rotated back to the initial position and the next trial commenced after the subjects were reminded to memorize their orientation.

Chair position and indicator setting were recorded (125 Hz) for further analysis. Statistical analysis was done on the final indicator orientation pooled for positive and negative final orientations using a repeated-measures ANOVA with two within-subjects factors: oriend.
Model simulations

To show the difference in expected responses for commutative versus noncommutative processing, we simulated both cases. As implementation of the noncommutative model, any mathematically correct transformation from angular velocity to angular orientation is possible. In the simulations shown, we used a rotation vector model. A rotation vector (underline indicates a vector) describes angular orientation using only three components (Haustein 1989)

\[ \varepsilon = \tan(\psi/2) \cdot \vec{n} \]  

where \( \psi \) is the angle of rotation around an axis \( \vec{n} \). The differential equation to compute the angular orientation \( \varepsilon \) from angular velocity \( \omega \) is given by (Hepp 1994)

\[ \varepsilon = [\omega + (\omega \cdot \varepsilon) \cdot \varepsilon + \omega \times \varepsilon]/2 \]  

where \( \circ \) denotes the inner or dot product and \( \times \) the vector cross-product. Equation 2 was used for the noncommutative simulations.

Because a commutative model is mathematically incorrect for arbitrary rotations, the model chosen was designed to fulfill two criteria: 1) The commutative model should be correct for rotations around a single axis and 2) the angular orientation resulting from the commutative model should be a rotation vector (Eq. 1) for easy comparison. This results in

\[ \varepsilon = \tan(1/2/H_\omega dt) \]  

Equation 3 was used for the commutative simulations.

Results

Model simulation

Figure 3 depicts the model simulations for the noncommutative model (filled symbols) and the commutative model (open symbols). As expected, the noncommutative model shows veridical responses, i.e., simulated perceived orientation coincides with physical orientation (dashed line). Note that, because of the path independence of rotations, final orientations for the three paradigms overlap. In contrast, the simulations of the commutative model, which simply integrates the velocity stimulus, show a clear dependency on paradigm: whereas simulated responses for roll-yaw and combined rotations are comparable, predicted responses for pitch-roll differ from these by about 80°.

Figure 4 shows the final indicator orientation for all ten subjects, rotational paths (circles: pitch-roll; squares: roll-yaw; diamonds: combined), and final orientations. Seven subjects for the three paradigms overlap. In contrast, the simulations of the commutative model, which simply integrates the velocity stimulus, show a clear dependency on paradigm: whereas simulated responses for roll-yaw and combined rotations are comparable, as expected from their similar velocity profiles (cf. Fig. 2, middle and bottom), predicted responses for pitch-roll differ from these by about 80°.
adjusted the indicator close to veridical responses (diagonal lines in Fig. 4), thus clearly showing a response that reflected their final orientation. The three remaining subjects differed in that they either responded as if they had been pitched back to the supine position only (S5, S7), i.e., they adjusted the line around 180°, or showed a mixture of responses (S8). S8 reported to be confused by the experiment and considered herself unable to solve the task. Another subject (S1) adjusted the indicator twice to the 180° position.

As evident from Fig. 4, most subjects’ responses clearly differed for final body orientations with positive or negative angles (final heading at about 90 or 270°) within the earth-horizontal plane. To test whether the small differences between final orientations on one side also resulted in different responses, we pooled positive and negative data from each subject. A first ANOVA including the data from all subjects showed no effect of final orientation or path. However, when subjects S5, S7, and S8 and the two outliers of S1 were excluded, a main effect of final orientation was found ($F(2,12) = 7.2, P = 0.009$), but no effect of path ($P > 0.7$) and no interaction ($P > 0.3$). The main effect of final orientation shows that, on average, subjects were aware even of the small differences in final self-orientation on one side and did not just always respond with the same indicator angle irrespective of end position (Fig. 5A). Errors in estimated orientation depended on the final chair orientation with smaller errors for the 100° end position [80°: 28.5° (SE 3.9°); 90°: 18.8° (SE 4.1°); 100°: 15.4° (SE 5.0°)].

A comparison of the experimental results with the model simulations (Fig. 5B) shows that errors in perceived final orientation are much larger than expected from a perfect noncommutative model, but did not depend on rotation sequence as expected from a commutative model.

**Discussion**

The majority of our subjects were able to estimate self-orientation in the horizontal plane after being rotated into a supine position along three different paths. The estimation errors show that subjects, on average, underestimated their angular position in the horizontal plane with respect to their original orientation. The comparison of model predictions for a mathematically correct model (noncommutative model; see methods) and a commutative model (see methods) shows that for the commutative model estimated final orientation in the horizontal plane differed by about 80° between the pitch-roll condition and the other two conditions observed. In fact, as shown in the **Appendix (Noncommutativity and path independence)**, any commutative processing of the successive rotations would imply that the perceived final orientations are different. Thus our subjects were able to take the noncommutativity of self-rotations into account, as reported previously for gaze stabilization by the vestibuloocular reflex (Tweed et al. 1999). Functionally, this remarkable ability may prove necessary to prevent falls when making rapid simultaneous or subsequent head and body movements. For example, bending forward by 90° while turning the head to finally look right or left does not normally cause spatial disorientation or falls.

Three of ten subjects (two females, one male) were not able to correctly indicate their self-orientation. Two subjects responded as if they had been tilted back by only 90°, but did so independently of condition (that is, even for roll-yaw, which...
involved no pitch rotation). One subject was confused; her responses varied between correct indication, mirror indication (side error), and pitch-back response. Because none of these subjects suffered from vestibular deficits, we hypothesize that their inability arises from a deficiency in the central processing of vestibular self-velocity information, such as in the ability to update self-orientation by 3D path integration.

The orientation estimates of the remaining subjects showed errors of about 20° on average. This raises the question of whether path-independent orientation estimates are compatible with such large estimation errors. An underestimation of angular velocity in one of the components (such as in roll) would lead to large differences between conditions (Fig. 5C, open symbols), i.e., to path-dependent responses even though the processing is still noncommutative (see the APPENDIX, Modified noncommutative models). However, in the present case, a solution that accounts for both path independence and estimation errors can be found as shown in Fig. 5C (filled symbols) by assuming differences in processing roll versus pitch and yaw information (see the APPENDIX, Modified noncommutative models). Thus the experimentally determined path independence and estimation errors are compatible with a noncommutative model. The weighting of the derivative of roll orientation used here is similar to that proposed earlier to explain the eye-position dependency of the torsional vestibuloocular reflex (Tweed 1997). For visuospatial updating, it was recently shown that the processing of roll and yaw rotations differs indeed with respect to processing of gravity as an additional cue (Klier et al. 2006).

Because 3D rotations on earth necessarily involve changes in orientation with respect to gravity, there is an alternative to the exclusive use of angular velocity information. Subjects could have used gravitational cues that are sensed by the otoliths, truncal graviceptors (Mittelstaedt 1996), or somato-sensation to determine their orientation with respect to gravity. However, note that the gravity cues available at the end of the rotation are not sufficient to determine the orientation in the earth-horizontal plane: subjects were in the supine position in all occasions and thus their final orientation with respect to gravity was always the same. Thus using gravity cues for this task also requires an ongoing updating process, i.e., keeping track of the change to the direction of gravity with respect to the head. Such a solution, if implemented correctly, allows accurate estimates of self-orientation and also involves non-commutative processing of both graviceptive and angular velocity cues (see the APPENDIX, Projection model; predictions of this “projection model” are equivalent to the noncommutative model shown in Fig. 5B). Using such processing, updating of the orientation in the earth-horizontal plane using only the changing gravity cues would be possible for the roll-yaw condition, but not for pitch-roll: because the second rotation in the pitch-roll condition was a turn around the earth-vertical axis (coinciding with the subject’s nasooccipital axis), it should have gone unnoticed. In contrast, roll-yaw consisted of two rotations around earth-horizontal axes, thus making it possible to infer the final orientation solely from keeping track of changing graviceptive cues.

The importance of gravitational cues for orienting movements toward remembered targets was emphasized by various recent studies (e.g., Klier et al. 2005, 2006; Prieur et al. 2005; Van Pelt et al. 2005). For example, subjects could make accurate saccades to remembered target locations after roll rotations around an earth-horizontal, but not around an earth-vertical axis (Klier et al. 2005). However, such a dissociation was not observed in the present experiments. Another study (Van Pelt et al. 2005) showed that, after intervening whole body roll rotations, errors in saccades to previously seen visual targets were not related to the intervening rotation in roll, but to body orientation with respect to gravity. This finding evidently points toward the use of gravitational cues in updating visual space, suggesting that the same may be true for estimation of self-orientation.

We conclude that whatever sensory input is used, the majority of subjects seem to be able to correctly integrate self-velocity or graviceptive cues into a consistent estimate of self-orientation in the horizontal plane, i.e., an estimate that is independent of the path taken to get to that orientation. As we have shown, such a path-independent estimate is possible only if the computation of self-orientation is noncommutative.

APPENDIX
Noncommutativity and path independence

Rotations are noncommutative, except in the one-dimensional case. This means, if two subsequent rotations are performed in reverse order, the final orientation of the rotated object will not be the same (e.g., Tweed et al. 1999).

Consequently, temporal integration of a time-varying angular velocity vector alone does not yield angular position, except in the one-dimensional case. This is because temporal integration can be expressed as the cumulative summation of the integrated variable over infinitely small time steps. Because summation is commutative, so is integration.

Thus for temporal integration to yield angular orientation, it is not angular velocity, but the derivative of angular orientation that needs to be integrated. This derivative is computed from angular velocity and orientation (e.g., Haslwanter 1995).

In the present experiments, two sequences of different rotations (pitch-roll and roll-yaw) were performed consecutively, yielding the same final orientation. During a particular rotation (e.g., roll), the orientation of the rotation axis with respect to the subject did not change, and thus each rotation was one-dimensional. Thus for each rotation alone, one could argue that a commutative operation would suffice to estimate the final orientation with respect to the starting orientation before this particular rotation. To answer the question of whether noncommutativity is known to the brain, the concatenation of the two rotations is crucial.

The final orientation $\mathbf{r}$ (underline indicates a vector) of the chair following the rotations $\mathbf{r}_2$ was

$$\mathbf{r} = \mathbf{r}_1 \otimes \mathbf{r}_2 = \mathbf{r}_1 \otimes \mathbf{r}_2$$

where $\otimes$ denotes the concatenation of two rotations, e.g., the product of two rotation matrices or two quaternions (e.g., Haslwanter 1995). Note that $\mathbf{r}_2$, the roll rotation, appears in both sequences of rotations because the rotating chair is a gimbal system in which the inner axis rotates the subject around the same subject-fixed roll axis in both sequences.

We will show now that, if the brain mistakenly assumed that rotations are commutative, the final orientation estimates would not be equal. That is, if the estimates of orientation for the same rotations, but in different order, were equal, then the estimates for the two sequences of different rotations shown earlier would not be the same.

In the following, the sign "\^" indicates such estimates from commutative computations. The estimate of orientation after the sequence $\mathbf{r}_2$ followed by $\mathbf{r}_1$ can thus be written as

$$\hat{\mathbf{r}}_{12} = \mathbf{r}_1 \oplus \mathbf{r}_2$$

where $\oplus$ denotes the concatenation of two rotations, e.g., the product of two rotation matrices or two quaternions (e.g., Haslwanter 1995). Note that $\mathbf{r}_2$, the roll rotation, appears in both sequences of rotations because the rotating chair is a gimbal system in which the inner axis rotates the subject around the same subject-fixed roll axis in both sequences.
and that of the second sequence

$$\tilde{L}_{3,2} = L_3 \oplus L_2$$

Here we used the symbol $\oplus$ to denote any possible commutative concatenation of rotations.

Because commutativity is assumed, the order of rotations does not matter, and thus

$$\tilde{L}_{2,3} = L_2 \oplus L_3$$

Thus both estimates $\tilde{L}_{1,2}$ and $\tilde{L}_{2,3}$ can be computed from rotation $L_2$ followed by a second rotation ($L_1$ or $L_3$, respectively). This, however, shows that the estimate $\tilde{L}_{2,3}$ could be equal to $\tilde{L}_{1,2}$, only if rotations $L_1$ and $L_3$ were equal or zero, or if the brain were mistakenly unable to distinguish between both. The first was not the case in the present experiment and the second can be safely excluded because it would mean that the brain cannot distinguish a 90° pitch-back rotation from a 90° yaw rotation. Thus assuming commutativity, the two sequences would yield different orientation estimates, as had to be shown.

Note that the proof makes no assumptions about how the integration of angular velocity or concatenation of successive rotations is actually performed by the brain, i.e., the experiment shows that the brain must necessarily take into account the noncommutativity of rotations to arrive at the same final estimate of orientation. This is also the case if the computations would be performed in an earth-fixed rather than a subject-centered coordinate system. To show this, one can simply translate the subject-centered rotations into earth-fixed rotations (pitch-roll translates to pitch-yaw, and roll-yaw translates to roll-pitch) and adopt the same line of reasoning as described earlier.

**Modified noncommutative models**

For simulations shown in Fig. 5C, the model in Eq. 2 (METHODS) was modified. For the underestimation of angular velocity (Fig. 3C, open symbols), the roll component of angular velocity was multiplied by 0.75.

For the simulations showing estimation errors similar to actual data with path independence (filled symbols), the model was modified as follows

$$\tilde{L} = G \cdot (\omega + (\omega \times \tilde{r}) \cdot \tilde{r} + \omega \times \tilde{r})/2$$

with

$$\tilde{r} = G^{-1} \cdot r$$

$$\begin{bmatrix}
0.5 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}$$

Thus the roll component of the derivative of orientation was weighted by 0.5 and the roll orientation used to compute the derivative was weighted by 0.2.

**Projection model**

Here, a mathematically correct model is derived for computing self-orientation in the earth-horizontal plane from two sensory inputs: angular velocity and gravity. Assuming that gravity $g$ is known from graviceptive sensory input or a central gravity estimator (e.g., Glasauer and Merfeld 1997; Merfeld and Zupan 2002), the $z$-axis of the space-fixed coordinate system $\tilde{z}$ can be expressed in subject-centered coordinates by

$$\tilde{z} = \frac{\tilde{g}}{||\tilde{g}||}$$

where $||$ expresses the norm of $\tilde{g}$. Thus $\tilde{z}$ is a unit vector parallel to gravity.

The space-fixed coordinate axis $\tilde{x}$ (within the earth-horizontal plane) expressed in subject-centered coordinates can then be computed from angular velocity $\omega$ as

$$\tilde{x} = \omega \times \tilde{x}$$

where $\times$ denotes the vector cross-product. By decomposing angular velocity $\omega$ into two components parallel and perpendicular to the z-axis, the $x$-axis can be expressed as

$$\tilde{x} = \omega_1 \times \tilde{x} + \omega_3 \times \tilde{x}$$

(A1)

The parallel component is determined by projecting the measured angular velocity vector $\omega$ onto the z-axis

$$\omega_3 = (\omega \times \tilde{z}) \cdot \tilde{z}$$

(A2)

where $\cdot$ denotes the inner or dot product. In contrast, the cross-product $\omega_1 \times \tilde{z}$, which points along the $z$-axis, can be expressed solely by using the gravity-derived $z$-axis and its temporal derivative

$$\omega_1 \times \tilde{z} = ([\omega \times \tilde{z}] \cdot \tilde{z}) \cdot \tilde{z} = (\omega \cdot \tilde{z}) \cdot \tilde{z}$$

(A3)

because $\tilde{z} = \omega \times \tilde{z}$. Inserting Eqs. A2 and A3 into Eq. A1 yields

$$\tilde{x} = (\omega \cdot \tilde{z}) \cdot \tilde{z} + (\omega \cdot \tilde{z}) \cdot \tilde{z}$$

(A4)

from which the instantaneous direction of the $x$-axis $\tilde{x}$ is determined. The first part of the equation uses angular velocity input from the semicircular canals, projects it onto the space-fixed axis parallel to gravity, and uses this earth-vertical component of angular velocity to derive that part of the change of self-orientation in the earth-horizontal plane, which is related to rotations within the earth-horizontal plane. The second part relies completely on graviceptive information to derive the part of self-orientation changes related to rotations around axes perpendicular to gravity. This model, which can be conceived as space-fixed computation of self-orientation, is noncommutative as well.

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